Evolutionary prisoner's dilemma game on hierarchical lattices

Jeromos Vukov
Department of Biological Physics, Eötvös University, H-1117 Budapest, Pázmány Péter sétány 1/A, Hungary

György Szabó
Research Institute for Technical Physics and Materials Science, P.O.Box 49, H-1525 Budapest, Hungary
(Dated: February 2, 2008)

An evolutionary prisoner's dilemma (PD) game is studied with players located on a hierarchical structure of layered square lattices. The players can follow two strategies [D (defector) and C (cooperator)] and their income comes from PD games with the "neighbors." The adoption of one of the neighboring strategies is allowed with a probability dependent on the payoff difference. Monte Carlo simulations are performed to study how the measure of cooperation is affected by the number of hierarchical levels (Q) and by the temptation to defect. According to the simulations the highest frequency of cooperation can be observed at the top level if the number of hierarchical levels is low ($Q < 4$). For larger Q, however, the highest frequency of cooperators occurs in the middle layers. The four-level hierarchical structure provides the highest average (total) income for the whole community.

PACS numbers: 89.65.-s, 02.50.Le, 87.23.Kg, 87.23.Ge

I. INTRODUCTION

The evolutionary prisoner's dilemma game (PDG) [1, 2, 3, 4, 5] is a very useful tool to examine the maintenance of cooperative behavior among selfish individuals. Investigating this model, we can study the behavior of a fictive society: we can collect data about the change of the total income when varying the model parameters, e.g., payoffs, set of strategies, topological structure of interaction, rules controlling the choice of a new strategy, noise, and external constraints.

The original PDG is a version of matrix games describing the interaction between two players. The payoffs of the players depend on their simultaneous decisions to cooperate or defect. In this paper we use rescaled payoff parameters without any loss of generality in the evolutionary PDGs [6]. Thus, mutual cooperation yields the players unit income providing them with the highest total payoff. On the contrary, their payoffs are zero for mutual defection. If one of the players defects while the other cooperates then the defector receives the highest individual payoff ($b > 1$), i.e., there is a temptation to defect, while the cooperator's income (called sucker's payoff) is the lowest one ($c < 0$) . In the PDG the cooperator's loss goes beyond the profit of the defector, i.e., $b + c < 2$. The values of these payoffs create an unresolvable dilemma for intelligent players who wish to maximize their own income; namely, defection brings higher individual income independently of the other player's decision but for mutual defection they receive the second worst result.

In the case of evolutionary multiagent systems [5, 7] we face a totally changed situation. The players' income is gained from iterated PDGs played with different coplayers. In the present study we handle only the two simplest strategies which are independent of the coplayers' decisions: the first one is always cooperating, while the second one always chooses defection. The players following these strategies are called cooperators (C) and defectors (D) respectively. In evolutionary games the players can adopt (learn) one of the more successful strategies from the coplayers they interact with. Generally, the probability of strategy adoption is determined by the payoff difference.

The mean field approximation predicts that in a society of players following the C or D strategy the cooperation dies out after a short time. This happens in the case of the one-dimensional chain if the interaction is limited to nearest neighbors. At the boundary of a C and D cluster the defectors always have higher payoff than the cooperators; thus cooperation vanishes quickly.

To demonstrate the advance of local interactions in the spatial games, Nowak and May [6] have created a two-dimensional cellular automaton where the players could follow the C or D strategy. Their investigations revealed that the cooperators overrun the territory of defectors along straight-line fronts while the defectors' invasion can be seen along irregular boundaries. It was shown that as a result of these invasion processes the coexistence of D and C strategies took place with a population ratio determined by the model parameters. Noisy effects make the boundaries irregular and give more chance for defection [8, 9]. Randomly chosen empty sites on the lattice further the maintenance of cooperation [8, 10] by holding up the spreading of defectors. These sites behave like defectors (for $c = 0$), but they cannot disperse; they are "sterile." Finally, it turned out that the short range interactions between the localized players favor the maintenance of cooperation under some conditions (e.g., for small b values) even if just the simplest strategies (C or D) are allowed [6, 8, 9, 11, 12, 13]. If b, the temptation to defect, exceeds a threshold value depending on the evolutionary rules, then the advantage of local interactions will not be enough to preserve cooperation, so defection overcomes.

Recently spatial PDGs have been studied on different social networks. First, different studies were performed to reveal what happens on small-world networks [14]. It is found that cooperation can be maintained on these networks in a wide range of parameters [15, 16, 17, 18].

Nowadays the research is concentrated on the revelation of those circumstances for which the total income of the soci-

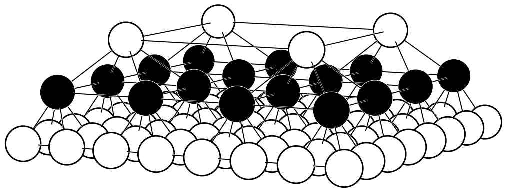

FIG. 1: Three-level "hierarchical lattice" constructed from square lattices (perspective view). Within a level the players are linked to their nearest neighbors, and to four other players located geometrically under them at the level below. The horizontal periodic boundary conditions are not indicated. White spheres indicate players at the first and the third layers; meanwhile black spheres are for players at the second level. In our notation the highest hierarchical level ($q = 1$) is at the top.

ety can reach its highest value. The other main purpose is to find networks describing social systems more and more adequately.

First we studied the PDG on such a scale-free hierarchical structure suggested by Ravasz and Barabási [19] where the distribution of the clustering coefficients is similar to that in social systems. The results of this model are enclosed in the Appendix.

As the cooperation is not maintained in this system, we modified the model's structure; we launched the study of a hierarchical structure with many "horizontal links." The equilibrium strategy concentrations and average payoffs are analyzed as functions of b for each hierarchical level. By this means we can get some information about those hierarchical structures providing the highest total payoff for such a society.

II. THE MODEL

We consider an evolutionary PDG with players located on the sites of a hierarchical structure (lattice). The three-level version of this structure is shown in Fig. 1.

The Q-level hierarchical structure is constructed from the sites of Q square lattices positioned above each other; meanwhile, the corresponding lattice constant is doubled level by level. These levels are labeled by $q = 1, \dots, Q$ from top to bottom. Within a given level the sites are linked (horizontally) to their four nearest neighbors. The undesired edge effects are eliminated by applying horizontal periodic boundary conditions for each level. In addition, the sites of the level q ($q = 2, \dots, Q$) are divided into 2×2 blocks whose four sites are linked to a common site above belonging to the level $q-1$. This means that for such a connectivity structure each player has four additional links to its "staff members" and another one to its "chief." Exceptions are naturally the players at the top ($q = 1$) and bottom ($q = Q$) levels where the "chiefs" and "staff members" are missing.

Consequently, the players at the top, middle, and bottom levels have eight, nine, and five neighbors, and the corresponding values of the clustering coefficient [20] are $1/7$, $1/6$, and $1/5$, respectively.

Note that this structure preserves the (horizontal) translational and rotational symmetry of the square lattice at the top level. For technical convenience, the horizontal sizes of the bottom layer are chosen to be $L = 2^k$ ($k > Q$) with a unit lattice constant. In this case this structure contains $N = 4[4^k - 4^{(k-Q)}]/3$ sites (players).

The players follow one of the above mentioned C or D strategies. The spatial distribution of strategies is described by a two-dimensional unit vector for each site x, namely,

$$s_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1}$$

for defectors and cooperators, respectively. Each player plays a PDG with its "neighbors" (coplayers) defined by the above structure and the incomes are summed. The total income of the player at the site x can be expressed as

$$M_x = \sum_{y \in \Omega_x} s_x^T A s_y \tag{2}$$

where the sum runs over all the neighboring sites of x (this set is indicated by Ω_x) and the payoff matrix has a rescaled form:

$$A = \begin{pmatrix} 0 & \mathrm{b} \\ \mathrm{c} & 1 \end{pmatrix}, \tag{3}$$

where $1 < b < 2 - c$ and $c < 0$ for the present PDGs. Since the work by Nowak and May [6] the parameter c is usually fixed to zero; therefore in our analysis we use the same value.

In the evolutionary games the players are allowed to adopt the strategy of one of their more successful neighbors. In the present work the success is measured by the ratio of total individual income and the number of neighbors (games), i.e.,

$$m_x = \frac{M_x}{|\Omega_x|}, \tag{4}$$

where $|\Omega_x|$ indicates the number of neighboring players at site x. This choice suppresses the advance coming from the larger number of neighbors.

In the present evolutionary procedure the randomly chosen player (x) can adopt one of the (randomly chosen) coplayer's (y) strategy with a probability depending on the difference of normalized payoff ($m_x - m_y$) as

$$W(s_x \leftarrow s_y) = \frac{1}{1 + \exp\left[(m_x - m_y)/T\right]}, \tag{5}$$

where T indicates the noise [7, 9]. This definition of W involves different effects (fluctuations in payoffs, errors in decision, individual trials, etc.). Henceforth, we consider the effects of Q and b on the measure of cooperation for $T = 0.02$.

III. AVERAGE STRATEGY FREQUENCIES AND PAYOFFS

The Monte Carlo (MC) simulations are carried out by varying the values of b and Q for fixed c and T values as mentioned above. For small system size, the MC simulations end up in

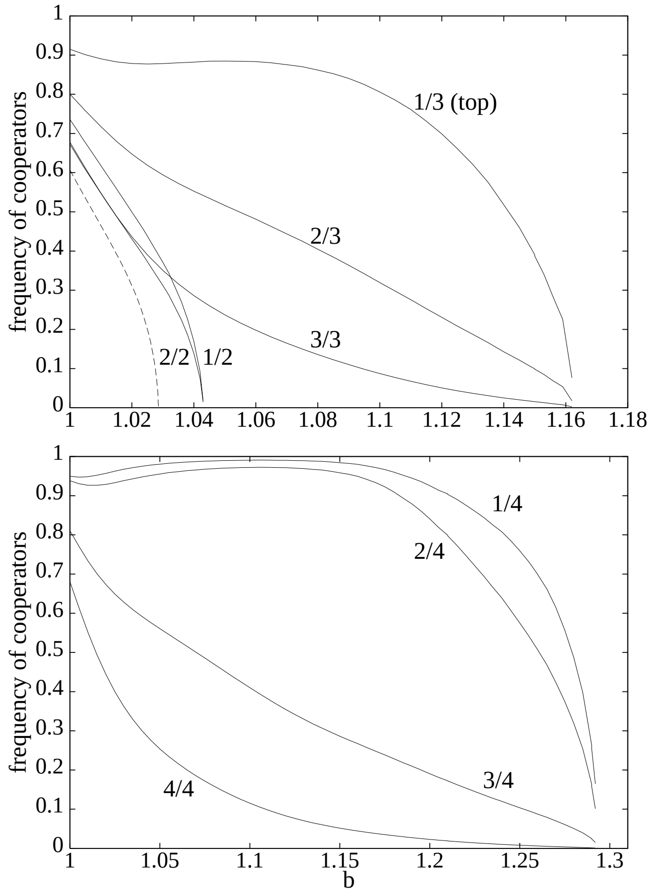

FIG. 2: Average frequency of cooperators as a function of b at different q levels for $Q = 2, 3$ (top), and 4 (bottom). The dashed line (top) indicates the results obtained on the square lattice ($q = Q = 1$).

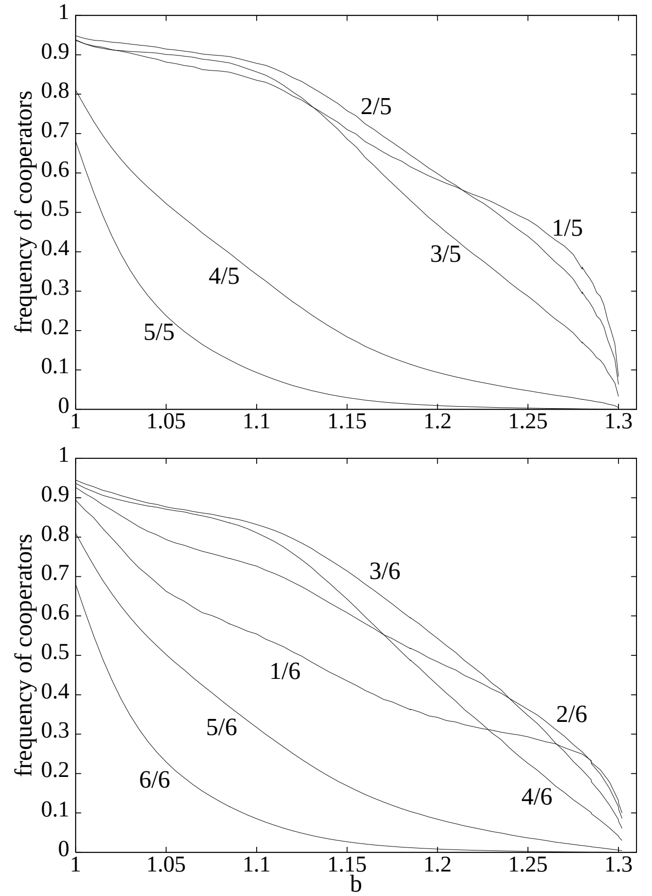

FIG. 3: The frequency of cooperators vs b on all the hierarchical levels for $Q = 5$ (top) and 6 (bottom).

one of the homogeneous absorbing states (homogeneous C or D phase) within a short time. In order to avoid this undesired phenomenon we have investigated sufficiently large systems (containing $N \simeq 10^5 - 10^6$ players) where the amplitudes of population fluctuations are considerably smaller than the corresponding average values.

The MC simulations are started from a random initial distribution of strategies where the average frequency of both strategies is 1/2. The above described evolutionary steps (strategy adoptions) are repeated until the system reaches the steady state we study. It is checked in some cases that the steady states are independent of the initial conditions. The steady states are reached after some transient time varied from 2000 to $50\ 000$ Monte Carlo steps (MCS), where during the time unit 1 MCS each player has a chance once on average to modify its strategy.

In the course of the simulations we have recorded the current frequencies [$\rho_D(t)$ and $\rho_C(t) = 1 - \rho_D(t)$] and payoffs [$m_D(t)$ and $m_C(t)$] as a function of time t for both strategies at each level. The corresponding average values are obtained by averaging through $16\ 000$ MCS. With knowledge of these quantities, we can determine other quantities concerning the whole system (total income of the society, strategy distributions referring to the whole system, etc.).

First we study how the strategy populations at each level depend on Q and b. The results for $Q = 2, 3$, and 4 are plotted in Fig. 2, and Fig. 3 represents the data for $Q = 5$ and 6. The corresponding system sizes are $N = 81\ 920$, $344\ 064$, $348\ 160$, $349\ 184$, and $349\ 440$. For these sizes the statistical error is comparable with the line thickness. It is observed that there always exists a threshold value (b_c) for any value of Q. For $b > b_c$ the cooperators become extinct. This critical value, however, depends on the number of hierarchical levels: $b_c(Q)$. The exact value of b_c for each Q is not determined because the thermalization time (as well as the fluctuation) grows very fast when approaching the critical points. Consequently, the precise determination of b_c requires a very long computer time (this is the reason why the end points are not represented in the figures). It is expected that the extinction of cooperators at large spatial scales belong to the two-dimensional directed percolation universality class [9, 21]. Unfortunately, the numerical justification of this conjecture exceeds our computer capacity.

We have performed MC simulations on the square lattice with periodic boundary conditions ($Q = 1$), too. The frequency of cooperators vanishes continuously as a function of b (dashed line in the upper plot of Fig. 2). The value of the

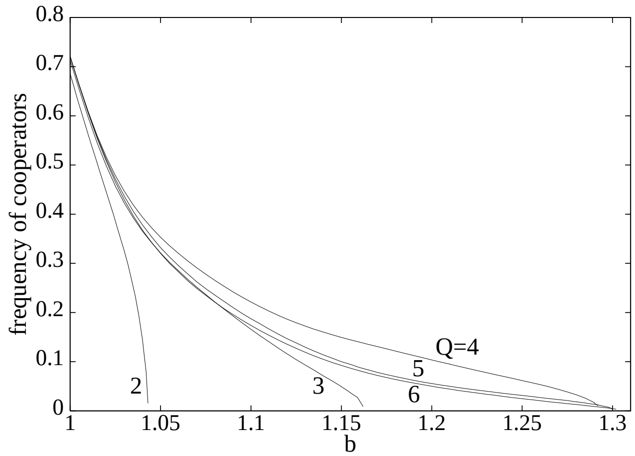

FIG. 4: The frequency of cooperators in the whole system as a function of b and Q.

critical point is $b_c = 1.028\,524(5)$. The phase transition belongs to the two-dimensional directed percolation universality class [9, 21].

Comparing the results of the two-level case with those on the square lattice one can see an increase of b_c that may have come from the enhancement in the average number of neighbors (from 4 to 5.6) and/or the average clustering coefficient (from 0 to 0.189). It is worth mentioning that both of these quantities increase monotonically with Q. In the limit $Q \to \infty$ the average number of neighbors goes to six and the average clustering coefficient tends to $23/120 = 0.191\dot{6}$.

Figures 2 and 3 show clearly that the frequencies of cooperators differ level by level. At the same time, the phase transitions to the absorbing (homogeneous) D state occur for each level at the same critical value of b dependent on Q. This feature is related to the fact that the successful colonies of cooperators can pass their strategy to any level although its probability depends on q.

Another striking message of the above numerical results is that the lowest measure of cooperation always occurs at the bottom level. The label of the most cooperative level, however, depends on Q and b. For $Q = 2, 3$, and 4 the frequency of cooperators is the highest at the top level, and it is decreasing monotonically when going downward on these hierarchical structures. The simulations for the given parameters show different behaviors if $Q > 4$. In the five-level system the highest measure of cooperation is reached by players in the second level in a wide range of b and their frequency is exceeded by the cooperators of the top level only in the vicinity of the critical point. The interval where the measure of cooperation is the highest at the top level becomes narrower for $Q = 6$. In this case one can observe two additional regions of b where the second and third levels exhibit the highest cooperativity. Furthermore, within a wide range of b the measure of cooperation at the fourth level is higher than it is at the top level.

Summarizing the data of the different levels, we can determine the measure of cooperation in the whole system as a function of Q and b. The average values are very low. The

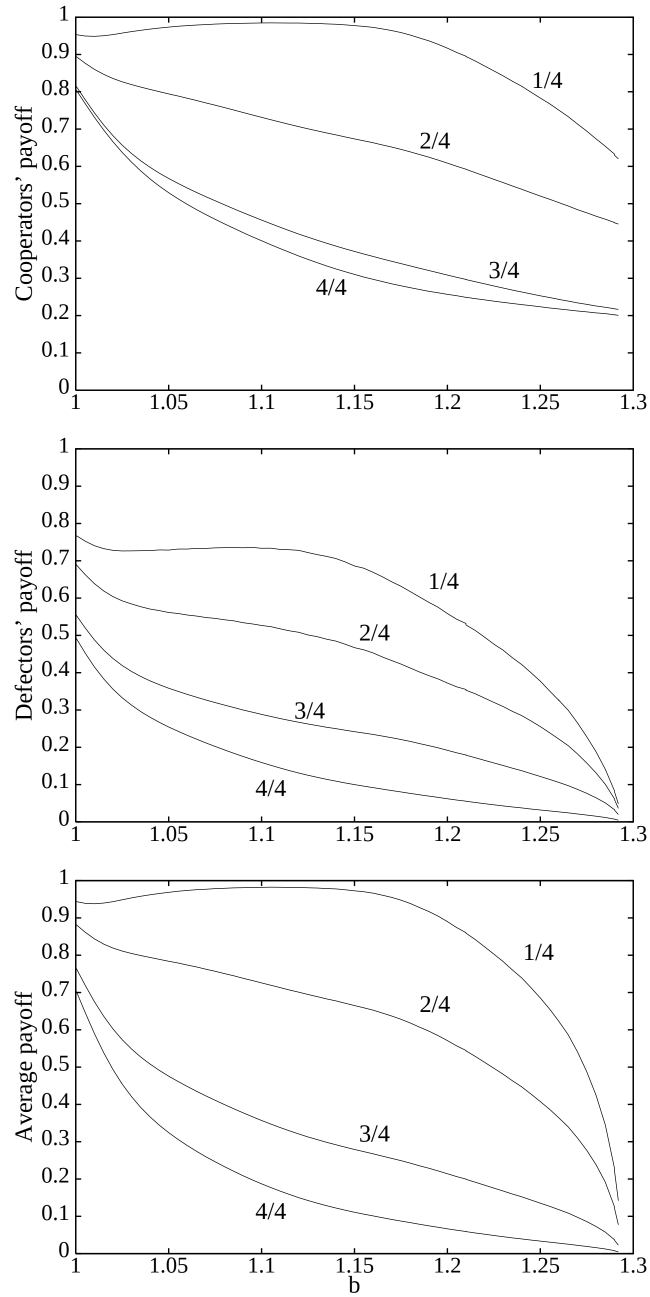

FIG. 5: The average (normalized) payoffs as a function of b at each hierarchical level for $Q = 4$. The top (middle) plot shows the average payoffs of cooperators (defectors) while the bottom figure indicates the weighted average payoffs at each level.

reason for this is that the measure of cooperation is the lowest at the bottom level where most of the players are located. As plotted in Fig. 4 the measure of cooperation has its maximum for $Q = 4$ for almost any value of b. It is observable that the $b_c(Q)$ values increase monotonically with Q, and they seem to converge to a value near $b_c(Q = 6)$. This tendency is likely to come from the mentioned convergence of the average number of neighbors and/or the average clustering coefficient.

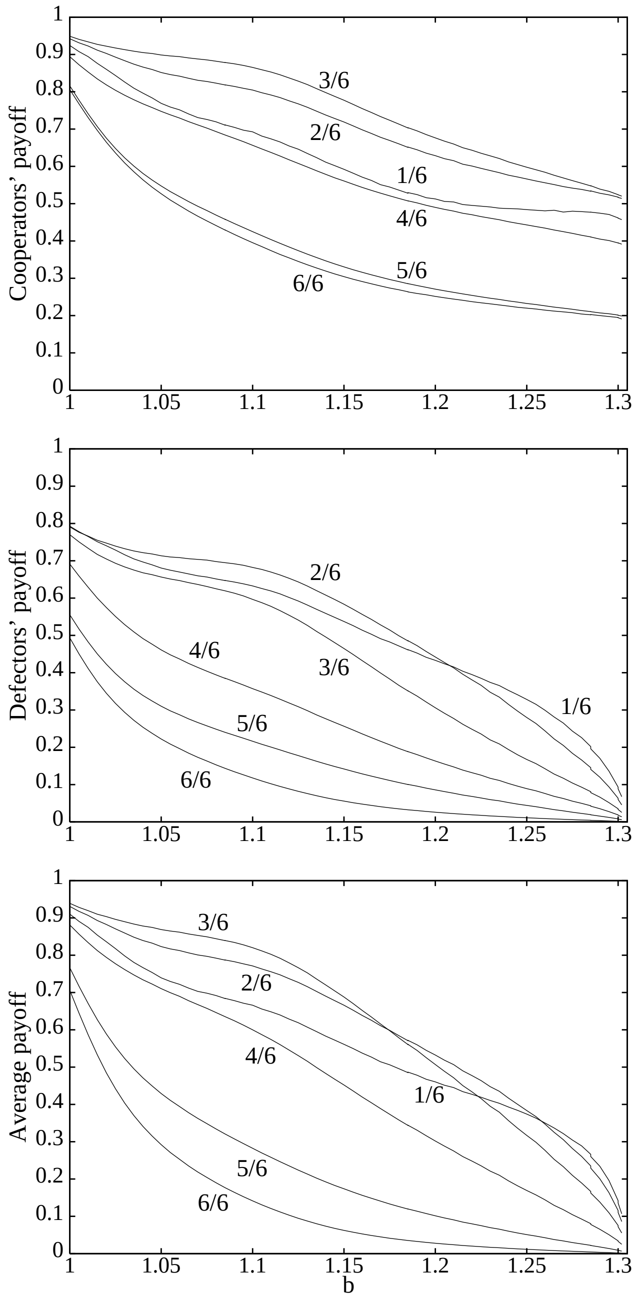

FIG. 6: The average (normalized) payoffs vs b at each hierarchical level for $Q = 6$.

Now we are going to analyze the average payoffs separately at the different levels as well as in the whole system. Figures 5 and 6 show the average payoffs of cooperators and defectors at each level for several Q values. Beyond that, the players' average payoff at each level is also presented. Comparing these figures with those representing the frequency of cooperators for the same Q, one can observe high similarity. This similarity is due to the fact that the cooperative partners ensure (positive) income for both the cooperators and defectors. (The total payoff for a player following the C or D strategy and having n neighbors following the C strategy is equal to n or bn, respectively.) Notice, furthermore, that the dominant part of the total income is received by the cooperators whose average incomes exceed those of defectors at each level. This fact comes from the formation of colonies by cooperators explaining why their average income remains positive for their vanishing concentrations. Although along the boundary of these colonies the defectors exploit cooperators, their average income is lower because most of them are surrounded by defectors and receive zero income. Evidently, the average payoff in the whole system (as well as the defectors' payoffs for each level) disappears continuously when approaching the critical point ($b \to b_c$) from below.

In the six-level system, the rank of the average defector payoff differs from that of the cooperator payoff. The average income of defectors is the highest at the second and the first levels. The reason for this is that for $Q = 6$, the highest measure of cooperation is found at the middle levels and as the result of the geometrical arrangement of the structure, these cooperators can be exploited the most by the defectors located geometrically above them.

For $Q = 2$ and 3 the average payoffs exhibit similar features as are found for $Q = 4$ (see Fig. 5) while for $Q = 5$ the system behavior is analogous to the case of the six-level system.

For the different Q values the b dependence of the total payoff per game (in the whole system) is very similar to the measure of cooperation plotted in Fig. 4. This means that the four-level hierarchy provides the highest average income to the players for such connectivity structures.

IV. SUMMARY

This work is devoted to studying the effect of the hierarchical structure on the measure of cooperation in a society where the pairwise interaction between the members is modeled by an evolutionary PDG with two strategies (C and D).

Using MC simulations we have determined the frequency of cooperators as a function of b (the measure of temptation to choose defection) at the different hierarchical levels in the Q-level structures. These systems exhibit a continuous (critical) transition from the $C + D$ state to the homogeneous D state if b tends to b_c depending on Q. During this transition the cooperators become extinct simultaneously at each level of the hierarchical structures.

The lowest measure of cooperation is always found at the bottom level. In the close vicinity of the transition point the frequency of cooperators is increasing gradually when going upward on the hierarchical structures. For lower b values, however, significantly different behavior is observed if $Q > 4$; namely, the highest frequency of cooperators is found in the middle levels; meanwhile the total payoff per game decreases if Q is increased for a fixed b. Surprisingly, the present simulations indicate clearly the existence of an optimum number of hierarchical levels where such a community can reach the highest income. For the present hierarchical lattices the four-level structure provides the highest income (productivity).

Another important message of the present investigations is related to the importance of “horizontal lattice structure” (or the additional horizontal links) in the hierarchical structures. As is mentioned in the Introduction, the cooperators die out exponentially in a similar evolutionary PDG model if the players are located on a complex hierarchical network suggested by Ravasz and Barabási [19]. We think that further systematic research is required to clarify the relationship between the maintenance of cooperation and the topological structure of connectivity.

Acknowledgments

This work was supported by the Hungarian National Research Fund under Grant No. T-47003.

APPENDIX: EVOLUTIONARY PRISONER’S DILEMMA GAME ON A SCALE-FREE HIERARCHICAL NETWORK

In this model, the players are located on a scale-free network, where the distribution of the clustering coefficients is similar to that in social systems. The dynamical rule in this model is the same as described in Sec. II; the difference occurs in the structure of the network.

The construction of the network suggested by Ravasz and Barabási [19] is the following. In the first step, there is a small cluster of five densely linked nodes [Fig. 7(a)]. In the next step, we generate four replicas of this cluster and connect the four external nodes of the replicated clusters to the central node of the old cluster, creating a large 25-node module [Fig. 7(b)]. Continuing the process, we again generate four replicas of this 25-node cluster, and connect the 16 peripheral nodes of each replica to the central node of the old module [Fig. 7(c)], obtaining a new module of 125 nodes. These replication and connection steps can be iterated indefinitely; in each step the number of nodes is multiplied by a factor of 5.

The generated graph has a power-law degree distribution with a degree exponent $\gamma = 1 + \ln 5/\ln 4 = 2.161$. In this network for $N \geq 125$ the average clustering coefficient is $C \simeq 0.743$. The network’s hierarchical structure is apparent: there are several, small, fully connected five-node graphs, which are clustered into larger 25-node graphs. The 25-node graphs are clearly separated from each other, and they are clustered into much larger 125-node graphs, etc.

The hierarchical levels are introduced in the following way. Being in the center of bigger cluster means being at higher level in the hierarchy. Figure 7(b) shows a three-level hierarchy: there is only one player at the first (highest) hierarchical level, the one in the center of the entire structure; the center nodes of the four “replicas” are at the second level, while the other nodes are at the third (lowest) level (there are 20 nodes of this type). One iterative step increases the number of hierarchical levels by 1. The analyzed network has $H = 7$ hierarchical levels; therefore the number of players (nodes) is $N = 5^{H-1} = 5^6 = 15\,625$.

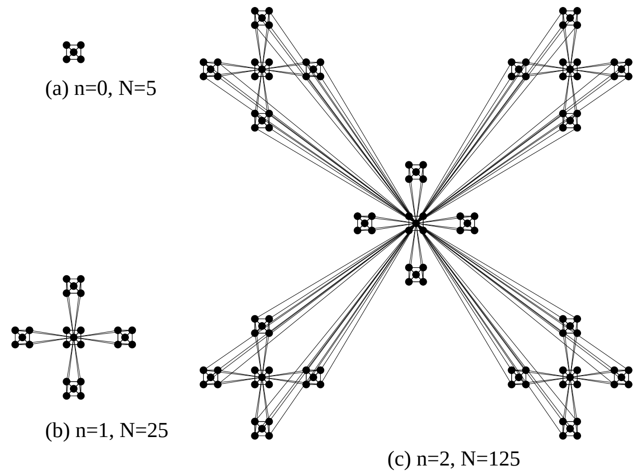

FIG. 7: Iterative steps (indicated by n) in the construction of the scale-free network.

We have performed Monte Carlo simulations as described previously by varying b for fixed temperature ($T = 0.01, 0.02$, and 0.05), to analyze what happens in a society where the players can follow C or D strategies.

Two different stages can be distinguished as the system evolves. The first one is a transient process, where most of the small groups of cooperators die out. After this period ($\simeq 4000$ MCS), the cooperators remain on such groups of sites where they can defend themselves against the defectors’ “attacks.” In this state the frequency of cooperators rather depends on the initial distribution of strategies than the value of b.

The small groups of cooperators can survive in two types of configurations. In the first case, they occupy a five-node subgraph [see the graph in Fig. 7(a)] with central node at the level $H-1$. The other basic unit of the C groups is composed of the four peripheral nodes of a five-node graph with a central node located at a hierarchical level $h \geq H - 1$.

Within such cliques [20], if all the players cooperate, then (due to the many internal links) they receive such a high income from each other that it provides protection against the external D invaders even for the highest b value. For this particular structure the normalized payoff of the attacking defectors is reduced drastically by the many neighboring D strategies. Occasionally, however, these groups can be invaded by defectors in the presence of noise. If a single D strategy gets into a clique then its offspring invade the whole group within a short time because these internal defectors (who have only a few defector neighbors) exploit all the internal cooperators.

The above process implies a decrease in the number of cooperators’ cliques as well as in the average frequency of cooperators. Figure 8 shows a noisy steplike decrease of C frequency in time where the magnitude of the steplike decrease corresponds to the disappearance of a cooperator clique described above. As the players in the small cooperator groups have higher average payoffs than the defectors, they invade continuously the territory of the defectors. At the same time, due to the network’s topology, the mutual cooperation cannot

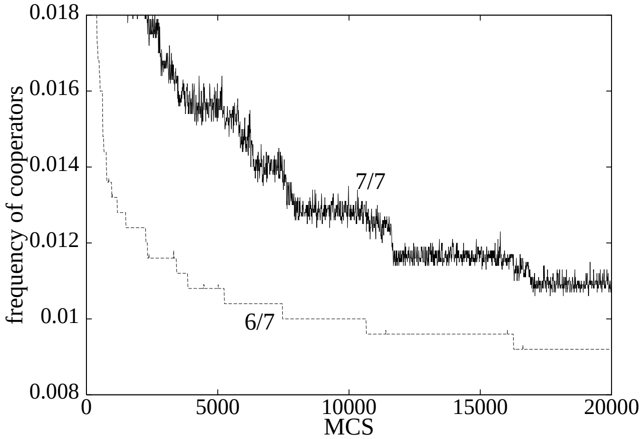

FIG. 8: The frequency of cooperators as a function of time at the two lowest hierarchical levels.

persist within these territories; therefore the D strategy will recapture them in a short time.

As a result of the above process the system will end up in the absorbing D state for any b. The average lifetime of the rare cooperator cliques depends on b and T. More precisely, the extinction of the cooperators becomes slower for lower b and T. The probability of the first successful D attack against a C clique can be estimated by the application of Eq. 5, and the corresponding theoretical predictions are consistent with the results of the simulations.

[1] J. Maynard Smith, *Evolution and the Theory of Games* (Cambridge University Press, Cambridge, U.K., 1982).
[2] J. Hofbauer and K. Sigmund, *Evolutionary Games and Population Dynamics* (Cambridge University Press, Cambridge, U.K., 1998).
[3] J. W. Weibull, *Evolutionary Game Theory* (MIT Press, Cambridge, MA, 1995).
[4] W. D. Hamilton, Am. Nat. **97**, 354 (1963).
[5] R. Axelrod, *The Evolution of Cooperation* (Basic Books, New York, 1984).
[6] M. A. Nowak and R. M. May, Int. J. Bifurcation Chaos Appl. Sci. Eng. **3**, 35 (1993).
[7] L. E. Blume, Games Econ. Behav. **5**, 387 (1993).
[8] M. A. Nowak, S. Bonhoeffer, and R. M. May, Int. J. Bifurcation Chaos Appl. Sci. Eng. **4**, 33 (1994).
[9] G. Szabó and C. Tőke, Phys. Rev. E **58**, 69 (1998).
[10] M. H. Vainstein and J. J. Arenzon, Phys. Rev. E **64**, 051905 (2001).
[11] F. Schweitzer, L. Behera, and H. Mühlenbein, Adv. Complex Syst. **5**, 269 (2002).
[12] M. Nakamaru, H. Matsuda, and Y. Iwasa, J. Theor. Biol. **184**, 65 (1997).
[13] C. Hauert, S. De Monte, J. Hofbauer, and K. Sigmund, J. Theor. Biol **218**, 187 (2002).
[14] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1998).
[15] G. Abramson and M. Kuperman, Phys. Rev. E **63**, 030901 (2001).
[16] B. J. Kim, A. Trusina, P. Holme, P. Minnhagen, J. S. Chung, and M. Y. Choi, Phys. Rev. E **66**, 021907 (2002).
[17] H. Ebel and S. Bornholdt, Phys. Rev. E **66**, 056118 (2002).
[18] G. Szabó and J. Vukov, Phys. Rev. E **69**, 036107 (2004).
[19] E. Ravasz and A.-L. Barabási, Phys. Rev. E **67**, 026112 (2003).
[20] B. Bollobás, *Modern Graph Theory* (Springer, New York, 1998).
[21] H. Hinrichsen, Adv. Phys. **49**, 815 (2000).